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Higher-Dimensional Vacuum Solutions of Einstein's Field Equations

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Some general solutions of the (general) D-dimensional vacuum Einstein field equations are obtained. The four-dimensional properties of matter are studied by investigating whether the higher-dimensional vacuum field equations reduce (formally) to Einstein's four-dimensional theory with matter. It is found that the solutions obtained give rise to an induced four-dimensional cosmological perfect fluid with a (physically reasonable) linear equation of state.

1. INTRODUCTION

Recently there have been many attempts to construct a unified theory based on the idea of a multidimensional spacetime (de Sabbata and Schmutzer, 1983; Applequist *et al.*, 1987; Collins *et al.*, 1989) and, indeed, it is generally believed that higher dimensions play a significant role in the early universe. Theories of this type date back to the original Kaluza–Klein theory (Kaluza, 1921; Klein, 1926*a*,*b*) in which the extra degrees of freedom in a five-dimensional theory were associated with an electromagnetic potential and the resulting Einstein field equations (EFE) mimicked the Einstein–Maxwell equations in four dimensions. A more recent approach to Kaluza–Klein-type models, which shall be referred to as the space-time-mass (STM) approach (Wesson, 1984, 1990), is via the interesting idea that fundamental physical constants (e.g., G, e, $\hbar$) can be used to construct new quantities (e.g., GMc^{-2}, $eG^{1/2}c^{-2}$, $\hbar^{1/2}G^{1/2}c^{-3/2}$) which can then be used as new coordinates in a higher-dimensional theory (Fukui, 1988; Coley, 1994) (in a similar fashion to the way c is used to construct a fourth coordinate ct in four-dimensional space-time theories).

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There are numerous examples of cosmological models of higher-dimensional theories [see, e.g., references in Bailin and Love (1987), Sokolowski and Golda (1987), and references cited above]. In this paper, we shall seek cosmological solutions of the higher-dimensional vacuum EFEs. Moreover, in the spirit of the original Kaluza–Klein theory (Einstein, 1956; Salam, 1980; Davidson and Owen, 1985), and consistent with the STM approach (Wesson, 1984, 1990; Coley, 1994), we shall investigate whether the properties of matter are completely geometrical in nature by studying whether the higher-dimensional vacuum EFEs reduce (formally) to Einstein's four-dimensional theory with a nonzero energy-momentum tensor constituting the material source. Assuming a ($D = 4 + N$)-dimensional Kaluza–Klein cosmology with product topology of the form

$$M^4 \times B^N \tag{1}$$

where M^4 is the four-dimensional spacetime with (zero-curvature) Robertson–Walker metric

$$g_{\mu\nu}^{\mathrm{RW}} = \mathrm{diag}(-1, R^2(t), R^2(t), R^2(t)) \tag{2}$$

with scale factor $R(t)$ (appropriate for the early universe), and assuming that the four-dimensional source is interpreted as a cosmological perfect fluid[2] with energy-density μ and pressure p, we find that the four-dimensional EFEs (with matter) then yield

$$\mu = 3\frac{\dot{R}^2}{R^2}$$

$$p = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} \tag{3}$$

It is known (Wesson, 1984, 1990; Coley, 1994) that in five dimensions the vacuum EFEs with internal scale factor $S(t)$ give rise [using (3)] to the familiar (zero-curvature) radiation Friedman–Robertson–Walker (FRW) model with

$$R = t^{1/2}, \qquad S = t^{-1/2}, \qquad \mu = \frac{3}{4}t^{-2} = 3p \tag{4}$$

In this model the properties of matter are prescribed entirely by the (five-dimensional) geometry; indeed, this approach gives rise to physically reason-

[2]It is known that the energy-momentum tensor of a particular form may formally admit a number of physical interpretations (Coley and Tupper, 1986). For example, in the original five-dimensional Kaluza–Klein theory the source was interpreted as an electromagnetic field and more generally in higher dimensions the four-dimensional source is assumed to be a (number of) scalar field(s) (Soleng, 1991).

able matter and a relevant cosmological model. Moreover, in five dimensions (4) is the unique solution whereby the extra dimension always compactifies.

We expect the theory to give rise to induced matter with very simple structure (Coley, 1994). However, it is of interest to study whether models exist with more complex forms of matter (i.e., more general equations of state) than that in (4). An investigation of the vacuum field equations of the general D-dimensional Einstein–Hilbert plus Gauss–Bonnet Lagrangian theory (Madore, 1985; Poisson, 1991) (in which the 'metric components' do not depend on the 'internal coordinates') reveals that ${}^{(4)}R \neq 0$ (and hence $\mu \neq 3p$) can only be achieved in higher dimensions ($D > 5$) or within generalized Lagrangian theories of gravity. Here we shall study the vacuum EFEs with general N (>1) internal dimensions.

2. ANALYSIS

We shall assume that the internal space B^N is the product of p compact Einstein spaces M^{N_i}; i.e.,

$$B^N = \prod_{i=1}^{p} M^{N_i}; \qquad \sum_{i=1}^{p} N_i = N \tag{5}$$

A. First let us consider the case $p = 1$, $B^N = S^N$, in which the D-dimensional metric is of the form

$$g_{ab} = (g_{\mu\nu}^{\mathrm{RW}}, S^2(t)h_{AB}) \tag{6}$$

where h_{AB} is the metric of the internal N-dimensional 'sphere' S^N of constant curvature K and 'radius' $S(t)$.

The nontrivial vacuum EFEs yield (Henriques, 1986; Wiltshire, 1987)

$$3\frac{\ddot{R}}{R} + N\frac{\ddot{S}}{S} = 0 \tag{7}$$

$$\frac{\ddot{R}}{R} + N\frac{\dot{R}}{R}\frac{\dot{S}}{S} + 2\frac{\dot{R}^2}{R^2} = 0 \tag{8}$$

$$\frac{\ddot{S}}{S} + 3\frac{\dot{R}\dot{S}}{RS} + (N-1)\left(\frac{K}{S^2} + \frac{\dot{S}^2}{S^2}\right) = 0 \tag{9}$$

From equations (7)–(9) we obtain

$$6\frac{\dot{R}^2}{R^2} + 6N\frac{\dot{R}}{R}\frac{\dot{S}}{S} + N(N-1)\left(\frac{K}{S^2} + \frac{\dot{S}^2}{S^2}\right) = 0 \tag{10}$$

If $S = S_0$ (constant), (7) and (8) yield $R = R_0$ (constant). When $R = R_0$, equations (3) imply $\mu = p = 0$. Henceforth we shall assume that $\dot{R} \neq 0$ and $\dot{S} \neq 0$. Equation (8) can then be integrated to yield

$$\frac{\dot{R}}{R} = \alpha S^{2N} R^{-3} \tag{11}$$

where α is a nonzero constant.

Assuming then that $K = 0$,[3] we can also integrate equation (9) to yield

$$\frac{\dot{S}}{S} = \alpha\beta S^{-N} R^{-3} \tag{12}$$

where β is a nonzero constant, and hence

$$\frac{\dot{S}}{S} = \beta \frac{\dot{R}}{R} \tag{13}$$

where, from equation (10), for $N \neq 1$ ($\beta = -1$ when $N = 1$)

$$\beta = \frac{-3N \pm (3N^2 + 6N)^{1/2}}{3N(N - 1)} \tag{14}$$

Equations (11) and (13) can then be used to obtain (after constant rescaling) the *general* solution

$$R = t^m; \qquad m = \frac{3 + \beta N}{3 + \beta^2 N} \tag{15}$$

$$S = t^n; \qquad n = \beta m \tag{16}$$

[and the *unique* solution (4) ($m = 1/2 = -n$) in the case $N = 1$].

From (14)–(16) we obtain

$$\begin{aligned} 3m + Nn &= 1 \\ 3m^2 + Nn^2 &= 1 \end{aligned} \tag{17}$$

Solutions of this type have been referred to as 'generalized Kasner solutions' in the literature (Henriques, 1986; Wiltshire, 1987).

From equation (3) we obtain

$$\begin{aligned} \mu &= 3m^2 t^{-2} \\ p &= m(2 - 3m)t^{-2} \end{aligned} \tag{18}$$

which implies the physically reasonable linear equation of state

[3]The case $K \neq 0$ is discussed in Coley (1994).

$$p = \sigma\mu; \qquad \sigma \equiv \frac{2 - 3m}{3m} \tag{19}$$

For each value of N (>1) there are two solutions [of (14)–(17)] for m and n of the form $m = m_+ > 0$ and $n = n_- < 0$ (which is the physically more interesting case in which the spatial dimensions expand while the internal dimensions compactify) and $m = m_- < 0$ and $n = n_+ > 0$. (Hence solutions need not compactify.) Corresponding to $m = m_+$, σ_+ decreases (from its largest value of 1/3 for $N = 1$) as N increases [toward a limiting value of $(2 - \sqrt{3})/\sqrt{3}$]. [Also, $n_-(n_+)$ increases (decreases) as N increases, etc.]

B. Next, partially motivated by the STM approach, we consider the case $p = N$, $B^N = \Pi_{i=1}^N M_i^1$, with the metric

$$g_{ab} = (g_{\mu\nu}^{\mathrm{RW}}, S_1^2(t), \ldots, S_p^2(t), \ldots, S_N^2(t)) \tag{20}$$

where the $S_i(t)$ are N cosmic scale factors.

The nontrivial vacuum EFEs then yield (no summation on i, j with $1 \leq i, j \leq N$)

$$3\frac{\dot{R}^2}{R^2} = -\sum_{i \neq j} \frac{\dot{S}_i}{S_i}\frac{\dot{S}_j}{S_j} - 3\frac{\dot{R}}{R}\sum \frac{\dot{S}_i}{S_i} \tag{21}$$

$$-2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = \sum \frac{\ddot{S}_i}{S_i} + \sum_{i \neq j} \frac{\dot{S}_i}{S_i}\frac{\dot{S}_j}{S_j} + 2\frac{\dot{R}}{R}\sum \frac{\dot{S}_i}{S_i} \tag{22}$$

and for each p,

$$1 \leq p \leq N: \qquad -3\frac{\ddot{R}}{R} - 3\frac{\dot{R}^2}{R^2} = \sum_{i \neq p} \frac{\ddot{S}_i}{S_i} + \sum_{i \neq j \neq p} \frac{\dot{S}_i}{S_i}\frac{\dot{S}_j}{S_j} + 3\frac{\dot{R}}{R}\sum_{i \neq p} \frac{\dot{S}_i}{S_i} \tag{23$\cdot p$}$$

From equations (23) we obtain

$$\text{(for all)} \quad 1 \leq i \neq j \leq N: \qquad \frac{\dot{S}_i}{S_i} - \frac{\dot{S}_j}{S_j} = \alpha_{ij} R^{-3} \prod_{k=1}^{N} S_k^{-1} \tag{24}$$

for an appropriately related set of integration constants α_{ij}. Integration then yields (after constant rescaling)

$$\text{(for all)} \quad 1 \leq i \neq j \neq k \leq N: \qquad S_k = S_i^{(1-\beta_{ijk})} S_j^{\beta_{ijk}} \tag{25}$$

for the constants β_{ijk} (related to the α_{ij}). Taking (21) $-$ 3 $\times$ (22) $+$ 2 $\times$ (23$\cdot p$) gives (for each p)

$$\frac{d}{dt}\left(\sum \frac{\dot{S}_i}{S_i} + 2\frac{\dot{S}_p}{S_p}\right) + \left(\sum \frac{\dot{S}_i}{S_i} + 3\frac{\dot{R}}{R}\right)\left(\sum \frac{\dot{S}_i}{S_i} + 2\frac{\dot{S}_p}{S}\right) = 0 \tag{26}$$

which implies that (for each p)

$$\sum \frac{\dot{S}_i}{S_i} + 2\frac{\dot{S}_p}{S_p} = \gamma_p R^{-3} \prod_{k=1}^{N} S_k^{-1} \tag{27}$$

(again the γ_p are integration constants). Integrating (and rescaling) then yields

$$\text{for each} \quad p, q \quad (p \neq q): \qquad S_p^{2\gamma_q(\gamma_p-\gamma_q)^{-1}} = \left(\prod_{k=1}^{N} S_k\right) S_q^{2\gamma_p(\gamma_p-\gamma_q)^{-1}} \tag{28}$$

which, in turn, implies that

$$\text{for all} \quad (1 \leq i \neq j \neq k \leq N): \qquad S_k = S_i^{(1-\delta_{ijk})} S_j^{\delta_{ijk}} \tag{29}$$

(where the δ_{ijk} are constants).

These equations then imply that for some p, say $p = 1$, for each i ($1 \leq i \leq N$) we can write (after rescaling)

$$S_i = S_1^{\epsilon_i} \tag{30}$$

for appropriate constants ϵ_i (i.e., each scale factor can be written as a power of S_1). From (24) it then follows that

$$S_1^{\left(\sum_{i=2}^{N} \epsilon_i\right)} \dot{S}_1 = \left(\frac{\alpha_{12}}{1 - \epsilon_2}\right) R^{-3} \tag{31}$$

It consequently follows from equation (21), using (30), that $\dot{R}R^{-1}$ is proportional to $\dot{S}_1 S_1^{-1}$, whence equation (31) leads to a power-law solution for $R(t)$, whence from (31) and then (30) we obtain power-law solutions for all the $S_i(t)$.

Therefore we have shown that the *general* solution of equations (21)–(23) is *necessarily* of the form

$$R = t^m; \qquad S_i = t^{n_i} \qquad (1 \leq i \leq N) \tag{32}$$

Substituting these expressions into equations (21)–(23) then yields the following algebraic constraints on the constants m, n_i;

$$3m + \sum_{i=1}^{N} n_i = 1 \tag{33}$$

$$3m^2 + \sum_{i=1}^{N} n_i^2 = 1$$

These solutions are consequently the analogs of the 'generalized Kasner solutions' [equations (15)–(17)] to which they reduce when each $n_i = n$.

Finally, equations (3) again yield $p = \sigma\mu$, where now the constant σ is a complicated expression in terms of the n_i. Also, there exist nontrivial solutions with $n_k = 0$ ($S_k = S_0$) for some value of k.

3. SUMMARY

We have studied higher-dimensional cosmological models by investigating vacuum solutions of the general D-dimensional EFEs. In the spirit of the original Kaluza–Klein theory, and consistent with the STM approach, the four-dimensional properties of matter are investigated by assuming that Einstein's four-dimensional theory with matter is embedded in a higher-dimensional theory.

We have obtained some new vacuum solutions of the higher-dimensional EFEs. For the (case A) D-dimensional cosmological line element (6) the general solution is given by equations (15)–(17), and for the (case B) metric (20) the general solution is given by equations (32)–(33). These higher-dimensional 'generalized Kasner' power-law solutions then give rise [through (3)] to a four-dimensional material consisting of a cosmological perfect fluid with linear equation of state [see equation (19)]. This induced material source is evidently physically reasonable and is somewhat more general than that of radiation, which is known to arise in the five-dimensional theory.

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